ON THE CALCULATION

OF THE VACUUM ENERGY DENSITY IN SIGMA MODELS

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A b s t r a c t

The vacuum energy density is calculated for the $O(N)$ nonlinear sigma models in two dimensions. To obtain ε_{vac} we assume that each point of the space in which non-perturbative fields are determined can be replaced by a sphere S^2 having a small radius r which approaches zero at the very end of the calculation. This assumption allows to get the classical fields generating v.e.v. of the trace of the energy-momentum tensor.

For a long time it has been clear that the critical phenomena in the theory of condensed matter and the quantum field theory are related to the conformal symmetry [1]. The symmetry breaking is defined by means of the trace of the energy-momentum tensor $\theta_{\mu\mu}$. Using $\theta_{\mu\mu}$ the phase structure of the σ-models in $2 \pm \varepsilon$ dimensions was investigated in [2]. The vacuum expectation value (v.e.v.) of $\theta_{\mu\mu}$ gives the vacuum energy density ε_{vac} and the characteristic mass scale for theories without dimension parameters [3].

ε_{vac} may be defined as

$$\varepsilon_{vac} = \frac{1}{D} < 0 \, |\theta_{\mu\mu}| \, 0 > \, ,$$

where D is the dimension of the space.

To obtain v.e.v. of the $\theta_{\mu\mu}$ we have to know the role of non-perturbative fluctuations in generating the physical amplitude. As a rule, the influence of instantons on v.e.v. of different operators is discussed because they are the sole non-perturbative fluctuations which are known in QCD. However, there is a set of models, such as the $O(N)$ nonlinear sigma models, which do not have instantons ($N > 3$), but do have the non-perturbative contributions [3, 4].

To understand the actual dynamics and the role of non-perturbative effects it is reasonable to study them on simple models.

For nonlinear sigma models some results were obtained in [5] in case of large N. In this paper it was shown that the non-perturbative contributions to the vacuum energy density are connected with the square of the symmetry current of the group $O(N)$. This result is valid for σ-fields, which are subject to the second class constraint.

The purpose of the present paper is to calculate ε_{vac} in the framework of the two - dimensional σ-models in the large N limit. To do this we should have an explicit form of the non-perturbative fields.

The physical idea of the calculation is based on the critical assumption that each point of the space, in which the non-perturbative fields are determined, can be replaced by a sphere S^2 having a small radius r which is set to zero at the very end of the calculation. This simple method allows to build non-perturbative fields in explicit form. From dimensional considerations ε_{vac} has to be of the order of $1/r^2$, therefore naive ε_{vac} goes to infinity when r tends to zero. Indeed, the value ε_{vac} is finite due to some quantum fluctuations which are orthogonal to classical fields. It will be shown that the quantum field correlator gives the value $exp(-4\pi/f)$ and thus we retrieve the familiar result [3]. Then non-perturbative part of the vacuum energy density ε_{vac}^n is written as

$$\varepsilon_{vac}^n = \frac{N}{8\pi} M^2 exp(-4\pi/f) \ .$$

We recall that the perturbative part of the vacuum energy density ε_{vac}^p is defined as

$$\varepsilon_{vac}^p = -\frac{N}{8\pi} M^2 \ ,$$

where M^2 is the regulator mass.

The Lagrangian of the $O(N)$ σ-models in 2 dimensions is taken in the

form

$$L = \frac{N}{2f} (\partial_\mu \sigma^a)^2 , \qquad\qquad a = 1, 2, ..., N \qquad (1)$$

and the N-component fields are subject to the constraint

$$\sigma^a \sigma^a = 1 . \qquad (2)$$

The constraint introduces the interaction between fields.

The action S_E for σ-models and the partition function $Z_E[\mathcal{J}^a]$ in euclidian space are the following:

$$Z_E[\mathcal{J}^a] = \int D\sigma(x) D\alpha(x) exp(-S_E + \int d^2x \mathcal{J}^a(x) \sigma^a(x)) ,$$

$$S_E = \frac{1}{2} \int d^2x \frac{N}{f} (\partial_\mu \sigma^a)^2 + \frac{\alpha(x)}{\sqrt{N}} (\sigma^a \sigma^a - 1) . \qquad (3)$$

The factor $N^{-1/2}$ under the Lagrange multiplier field $\alpha(x)$ is written for the sake of convenience.

In order to set off the non-perturbative contributions in the vacuum energy density one decomposes $\sigma^a(x)$ and $\alpha(x)$ as

$$\sigma^a = C^a(x) + q^a(x) ,$$

$$\alpha(x) = \alpha_C(x) + \alpha_{qu}(x) ,$$

where $C^a(x)$ and $\alpha_C(x)$ are classical fields, while $q^a(x)$ and $\alpha_{qu}(x)$ describe small fluctuations around the classical background. Under this decomposition in the large N limit, we get

$$Z_E = exp \left(-S_{cl} + \int d^2x \mathcal{J}^a(x) C^a(x) \right) \int Dq^a(x) D\alpha_{qu}(x)$$

$$exp \left(-S_{qu} + \int d^2x \mathcal{J}^a(x) q^a(x) \right) . \qquad (4)$$

Here,

$$S_{cl} = \frac{1}{2} \frac{N}{f} \int d^2x (\partial_\mu C^a)^2 ,$$

$$S_{qu} = \frac{1}{2} \int d^2x \frac{N}{f} (\partial_\mu q^a)^2 + \frac{\alpha_C(x) q^a q^a}{\sqrt{N}} + \frac{\alpha_{qu}}{\sqrt{N}} (C^a q^a + q^a q^a) .$$

The equations of motion for the $c^a(x)$ and $\alpha_C(x)$ fields are

$$\frac{N}{f} \partial^2 C^a = \frac{\alpha_C C^a}{\sqrt{N}} - \mathcal{J}^a ,$$

$$C^a C^a = 1 \qquad (5)$$

3

and for the quantum fields

$$2C^a q^a + q^a q^a = 0 \ ,$$

$$\tfrac{N}{f}\partial^2 q^a = \tfrac{\alpha_C q^a}{\sqrt{N}} - \mathcal{J}^a + \tfrac{\alpha_{qu}}{\sqrt{N}}(C^a + q^a) \ . \tag{6}$$

Let us simplify eqs.(6) using them in the linear form with respect to the quantum fluctuations. We have

$$C^a q^a = 0 \ ,$$

$$\tfrac{N}{f}\partial^2 q^a = \tfrac{\alpha_C}{\sqrt{N}}q^a + \tfrac{\alpha_{qu} C^a}{\sqrt{N}} \ . \tag{7}$$

We are interested in v.e.v. of $\theta_{\mu\mu}$, therefore $\mathcal{J}^a$ is taken equal to zero. Besides, the integral over α_{qu} in eq.(4) can be replaced by $\delta(C^a q^a)$, that is we are integrating over q^a being subject to the constraint (7).

It is known that v.e.v. of $\theta_{\mu\mu}$ can be obtained by varying $ln Z_E$ according to the change of the coupling constant on $\delta f = \xi \beta(f)$, where ξ - is the parameter of the global scale transformation and $\beta(f) = -M^2 \frac{df}{dM^2}$ [3, 6]

$$< 0|\theta_{\mu\mu}|0 >= \tfrac{\beta(f)}{2f^2}N \ < 0|\left(\partial_\mu q^a\right)^2|0 > \ ,$$

$$< 0|\left(\partial_\mu q^a\right)^2|0 > \ = \ Z_E^{-1} exp(-S_{cl}) \int Dq^a \delta(c^a q^a)(\partial_\mu q^a)^2 exp(-S_{qu}) \ . \tag{8}$$

Here, $S_{qu} = \tfrac{1}{2}\int d^2 x \tfrac{N}{f}(\partial_\mu q^a)^2 + \tfrac{\alpha_C}{\sqrt{N}}q^a q^a$.

Notice that we omit the term which is obtained by varying the classical action. As we shall see later, it will give the contribution to the pertiturbative part of the vacuum energy.

The problem of the calculation of the quantum correlation $< (\partial_\mu q^a)^2 >$ is to take into account the condition $C^a q^a = 0$ and to do a regularization of the quantum correlation.

The condition of orthogonality of quantum and classical fields can be satisfied by using the identity

$$\partial_\mu C^a = -J_\mu^{ab} C^b \ . \tag{9}$$

Here, $J_\mu^{ab} = C^a \partial_\mu C^b - \partial_\mu C^a C^b$. This is true when the fields are subject to the constraint $C^a C^a = 1$. Differentiating $C^a q^a = 0$ and using (9) we obtain

$$C^a \left(\partial_\mu q^a + J_\mu^{ab} q^b\right) \ = \ 0 \ . \tag{10}$$

For arbitrary C^a, eq.(10) is satisfied if

$$\partial_\mu q^a + J_\mu^{ab} q^b \ = \ 0 \qquad or \tag{11}$$

4

$$q^a(x) = Pexp\left(-\int_0^x dz^\mu \hat{J}_\mu\right)_{ab} q^b(0) .$$

Hence, $q^a(x)q^a(x) = q^a(0)q^a(0)$.

Then we can obtain

$$< 0| \left(\partial_\mu q^a\right)^2 |0 > = J_\mu^{ak} J_\mu^{ap} < 0|q^k(x)q^p(x)|0 >$$

and with due regard for

$$< 0|q^k(x)q^p(x)|0 > = \delta^{pk} < 0|q^i(x)q^i(x)|0 > ,$$

we get

$$< 0| \left(\partial_\mu q^a\right)^2 |0 > = 2\left(\partial_\mu C^a\right)^2 < 0|q^i(x)q^i(x)|0 > . \tag{12}$$

Here use was made of the identity $\left(J_\mu^{ab}(x)\right)^2 = 2\left(\partial_\mu C^a(x)\right)^2$. There is no sum on the repeated indices. We also suppose that for any indices $i = 1, 2, ..., N$ all correlators $< 0|q^i(x)q^i(x)|0 >$ are the same.

From eqs.(8) and (12) we get

$$< 0|\theta_{\mu\mu}|0 >= \frac{\beta(f)}{f^2}N\left(\partial_\mu C^a(x)\right)^2 < 0|q^i(x)q^i(x)|0 > . \tag{13}$$

In eq.(13) the magnitude of $< 0|\theta_{\mu\mu}|0 >$ is constant, at the same time the quantity $\left(\partial_\mu C^a(x)\right)^2$ is a function of x in a general case. In such case we have to find particular classical fields $C^a(x)$ which satisfy the condition $\left(\partial_\mu C^a\right)^2 = const$ and the constraint (5).

The first condition may be written in the form

$$\partial^2 C^a(x) = constC^a(x) . \tag{14}$$

To find some non-trivial solution to eq.(14) with account of (5) let us consider the special case when $N = 3$ and each point $x = x_0$ in the two-dimensional euclidean space, in which eq.(14) is defined, is replaced by a sphere S^2 with a small radius r in eucludean 3 dimensional space and center in x_0.

The quantity r is a new parameter which is set to zero at the very end of the calculation. From dimension considerations, eq.(14) must be written as

$$\partial^2 C^a = \frac{2C^a}{r^2} . \tag{15}$$

It can be verified that the solution of eq.(15) is the function $C^a = \frac{z^a}{r}$, $z^a = (x - x_0)^a$, $a = 1, 2, 3$, $r^2 = (z^a)^2$. So the solution coincides with the asymptotics of the Higgs fields for the monopoles of 't Hooft and Polyakov [7, 8]. The asymptotics of vector fields for the monopoles also coincides with the symmetry current in our models. In addition, the identity (9), in this case, coincides with the condition of the parallel transport of the Higgs fields along way dx, i.e. with the asymptotics of the covariant derivative. It may well be monopole of 't Hooft-Polyakov is defined in a sphere S^2 which has asymptotics at the distance r. It is not surprising because introducing of sphere S^2 in euclidean space is associated with the emergence of a non-trivial second homotopy group $\pi_2(O(3)) = Z_2$.

To obtain a solution of eq.(15) for any $N > 3$ we can choose the matrix of C-fields in such a way that the first three fields $C^i(i = 1, 2, 3)$ coincide with the solution for $O(3)$-group and the rest of N- fields is some constants. These C-fields have to be subject to the constraint (5) and therefore the fields C^i have to be normalized as $C^i C^i = g$. The multitude of the other special solutions can be obtained by rotating the solution in global $O(N)$ group.

We have obtained that the term $(\partial_\mu C^a)^2$ is of the order of $\mu^2 = 1/r^2$. Therefore the varying of the classical action determines the perturbative part of the vacuum energy.

To calculate the magnitude $< 0|q^i(x_0)q^i(x_0)|0 >$ we have to regularize it by separating the quantum fields in different points. That is, instead of the magnitude $< 0|q^i(x_0)q^i(x_0)|0 >$ we have to calculate the value

$$\lim_{\Delta \to 0} < 0|q^i(x_0 - \Delta)q^i(x_0 + \Delta)|0 > \ .$$

Using eq.(11) we obtain

$$\lim_{\Delta \to 0} < 0|q^i(x_0 - \Delta)q^i(x_0 + \Delta)|0 > = < 0|q^k(0)q^k(0)|0 > \lim_{\Delta \to 0} \left(- \int_{x - \Delta}^{x + \Delta} dx_\mu \hat{J}_\mu \right)_{ik} =$$

$$= < 0|q^k(0)q^k(0)|0 > exp \left(-2 \oint dx_\mu \hat{J}_\nu \right)_{ik} \ .$$

The coefficient 2 emerges at the circled integral because there are two different types of the circles. One of them is passed clockwise and the other is passed in the opposite direction. We sum over this types of the circles. The circled integral is $\oint dx_\mu \hat{J}_\nu = \oint dC^K C^i - dC^i C^K = 2S_{ik}$. The

area S_{ik} is enclosed by the circle which is oriented normally to the third axis in the iso-space and equals πg.

Let us denote the constant $< 0|q^i(0)q^i(0)|0 >$ by γ. We arrive at

$$< 0|\theta_{\mu\mu}|0 > \ = \ g\frac{\beta(f)}{f^2}N\frac{1}{r^2}\gamma e^{-4\pi g} \ . \tag{16}$$

Here, our result is defined on the scaling mass $\mu^2 = 1/r^2$. The constant γ is proportional to $\hbar$ and therefore ε_{vac} approaches zero in the classical limit ($\hbar \to 0$) as it must be for the quantum conformal anomaly.

In order to obtain the net result we ought to obtain $\beta(f)$ and to get the v.e.v. of $\theta_{\mu\mu}$ corresponding to $r^2 = 0$. Notice that the classical fields C^i defined on the scaling mass μ^2 can be written in terms of the bare classical fields C_0^i as $C^i = Z^{1/2}C_0^i$. In this case the relation between g and unrenormalized constant g_0 is written as $g = Zg_0$. We also define relation $< 0|\theta_{\mu\mu}|0 >= Z < |\theta_{\mu\mu}|0 >_0$.

In order to find Z we must get renormalized action S_r. In the large N limit the leading graph renormalizing the classical action is the tadpole diagram in which we integrate over momentum $p^2 \geq \mu^2$.

Thus we get

$$S_r = \int d^2x\frac{N}{f}(\partial_\mu C_0^i)^2(1 - \frac{f}{4\pi}\ln\frac{M^2}{\mu^2}) \ . \tag{17}$$

We recall that only the first three of C-fields are function of x. Therefore $(\partial_\mu C^a)^2 = (\partial_\mu C^i)^2, i = 1, 2, 3; a = 1, 2, ..., N$.

From eq.(17) we find

$$Z = 1 - \frac{f}{4\pi}\ln\frac{M^2}{\mu^2} \ . \tag{18}$$

There is minimum of the action under condition $\mu^2 = m^2$, where m^2 satisfies the equation

$$\frac{f}{4\pi}ln\frac{M^2}{m^2} \ = \ 1 \ . \tag{19}$$

Taking into consideration this equation we obtain

$$\frac{\beta(f)}{f^2} \ = \ \frac{1}{4\pi} \ .$$

Now, we get the final result

$$< 0|\theta_{\mu\mu}|0 >_0= \gamma g_0\frac{N}{4\pi}\mu^2 exp\left(-4\pi g_0\left(1 - \frac{f}{4\pi}\ln\frac{M^2}{\mu^2}\right)\right) \ . \tag{20}$$

We can see that this quantity is scaling independent only if $fg_0 = 1$. This means that the quantity g_0 is not a free parameter. When $\mu^2 = M^2$, we obtain

$$\varepsilon_{vac} = \gamma g_0 \frac{N}{8\pi} M^2 exp(-4\pi/f) \ . \tag{21}$$

The quantity ε_{vac} differs from the familiar result [3] by the coefficient γg_0. Therefore we may suppose that there is only part of ε_{vac} in eq.(21) which is defined by the monopole of 't-Hooft-Polyakov.

We have studied the contribution of the non-perturbative fluctuation in the vacuum energy density. It was shown that some classical fields were obtained provided that each point of the euclidean space has been replaced by sphere S^2 with a small radius. This method was associated with the emergence of the monopole in each such sphere because of the existence of the non-trivial map $S^2 \to S^2$.

Due to the quantum fields which are normal to the non-perturbative fields ε_{vac}^n remains finite when the sphere radius tends to zero.

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